Global persistence exponent in critical dynamics: Finite-size-induced crossover

D. Chakraborty* and J. K. Bhattacharjee†
Department of Theoretical Physics, Indian Association for the Cultivation of Science, Jadavpur, Kolkata-700032, India
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We extend the definition of a global order parameter to the case of a critical system confined between two infinite parallel plates separated by a distance L. For a quench to the critical point we study the persistence property of the global order parameter and show that there is a crossover behavior characterized by a nonuniversal exponent which depends on the ratio of the system size to a dynamic length scale.

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Global persistence exponent for nonequilibrium critical dynamics was introduced a decade ago [1], following the emergence of similar exponents in the evolution of Ising spins [2–4] in one and higher dimensions and the evolution of a diffusing field [5] from random initial conditions in different dimensions. The simplest system exhibiting persistence is the random walk in one dimension [6]. Since Brownian motion under restrictive geometry has been of experimental interest lately [7], the persistence problem was addressed under those situations [8]. It was seen that the power-law decay for the infinite system acquired an exponential correction for the confined system (confinement by walls or harmonic forces). This was in contrast to the finite persistence probability observed by Manoj and Ray for finite-size systems exhibiting critical dynamics. The quench carried out by Manoj and Ray [9] was, however, deep into the ordered region. For a D-dimensional Ising model, starting from a random initial condition, they quenched the system to $T=0$ and allowed the spins to evolve according to Glauber dynamics. Domains began forming and when it happened that the domain size became larger than the system size, then the persistence probability attained a finite value. The global persistence exponent of Majumdar *et al.* [1] was defined differently. It referred to the quench from a high temperature to $T=T_c$, the critical point of the system, and considered the global order parameter. The individual spins flip rapidly and the probability of not flipping in an interval has an exponential tail. It is only when the global order parameter is considered that one finds the power-law tail. In this situation if we consider a finite-size system [10–12], then for a sufficiently small system size (smaller than the appropriate "dynamical" length scale), the global order parameter will no longer be so difficult to "overturn" and an exponential tail could be expected just as it happened with the Brownian motion in restrictive geometry. In this paper we use the spherical limit to establish our result. Very recently Gambassi *et al.* have addressed a crossover in global persistence probability with initial finite magnetization [13].

We consider the usual Landau Ginzburg free energy F for the N-component order parameter ϕ_i $\{i=1,2,\ldots.N\}$, in a three-dimensional (3D) space, that is,

$$F=\int d^3\vec{x}\left(\frac{r}{2}\phi_i\phi_i+\frac{1}{2}(\nabla_j\phi_i)(\nabla_j\phi_i)+\frac{u}{4N}(\phi_i\phi_i)(\phi_j\phi_j)\right), \tag{1}$$

where the summation convention is implied.

The corresponding Langevin equation is given by

$$\dot{\phi}_i=\Gamma\nabla^2\phi_i-\Gamma\left(r\phi_i+\frac{u}{N}\phi_i\sum_j\phi_j^2\right)+\xi_i, \tag{2}$$

where ξ is a Gaussian white noise having correlation

$$\langle\xi(\vec{r},t)\xi(\vec{r'},t')\rangle=2\Gamma\delta(\vec{r}-\vec{r}')\delta(t-t'). \tag{3}$$

Since we will be using spherical limit, it makes sense to work in $D=3$ directly. The range of validity of the spherical approximation is for $2<D<4$ and hence $D=3$ is the natural choice.

The confinement is taken to be in the z direction and the orthogonal space has two dimensions. The confining is in the form of two "parallel plates" at $z=0$ and at $z=L$, where Dirichlet boundary conditions hold. The other two dimensions are infinitely extended. The decomposition of $\phi_i(\vec{r},t)$ is now in terms of Fourier transform in two dimensions and a Fourier series in the z direction, so that

$$\phi_i(\vec{r},t)=\int\frac{d^2\vec{k}}{(2\pi)^2}\sum_{n=1}^{\infty}\phi_{i,n}(\vec{k},t)\sin\left(\frac{n\pi z}{L}\right), \tag{4}$$

and the linearized Langevin equation becomes

$$\dot{\phi}_{i,n}(\vec{k},t)=-\Gamma\left(k^2+\frac{n^2\pi^2}{L^2}\right)\phi_{i,n}(\vec{k})-\Gamma r\phi_{i,n}(\vec{k})+\xi_i(\vec{k}), \tag{5}$$

in the noninteracting limit, $u=0$. The subscripts (i,n) are, respectively, component and mode indices. For the choice of $n=1$, $r=-\frac{\pi^2}{L^2}$, $\vec{k}=0$ gives us

$$\dot{\phi}_{i,1}(0,t)=\xi. \tag{6}$$

At the critical point for the confined system ($r=-\pi^2/L^2$ represents the mean field expression of the critical point), the lowest mode ($k=0$, $n=1$) undergoes a Brownian motion, corresponding to a persistence exponent $\theta=0.5$. For the finite-size system, we identify $\phi_{i,1}(0)$ as the global order parameter.

To work in the spherical limit we write Eq. (2), which is the original equation without the confining geometry, as

*tpdc2@mahendra.iacs.res.in
†tpjkb@mahendra.iacs.res.in

$$\dot{\phi}_i(\vec{k},t) = \Gamma\nabla^2\phi_i(\vec{k},t) - \Gamma r\phi_i(\vec{k},t) - \frac{\Gamma u}{N}\phi_i(\vec{k},t)\left(\left\langle\sum_{j=1}^{N}\phi_j^2\right\rangle\right) + \frac{\Gamma u}{N}\phi_i(\vec{k},t)\left(\left\langle\sum_{j=1}^{N}\phi_j^2\right\rangle - \sum_{j=1}^{N}\phi_j^2\right) + \xi_i(\vec{k},t).$$

Taking $\langle\phi^2\rangle=\langle\phi_j^2\rangle$ independent of j, we have

$$\dot{\phi}_i(\vec{k},t) = \Gamma\nabla^2\phi_i(\vec{k},t) - \Gamma(r+u\langle\phi^2\rangle)\phi_i(\vec{k},t) + \frac{\Gamma u}{N}\phi_i(\vec{k},t) \times\left[N\langle\phi^2\rangle - \sum_{j=1}^{N}\phi_j^2\right] + \xi_i(\vec{k},t). \tag{7}$$

Since $(N\langle\phi^2\rangle-\Sigma_j\phi_j^2)$ is of $O(1)$ we find in the limit $N\to\infty$ (spherical limit), for any i, (in momentum space)

$$\dot{\phi}_n(\vec{k},t) = -\Gamma\left(k^2 + \frac{n^2\pi^2}{L^2}\right)\phi_n(\vec{k}) + a(t)\phi_n(\vec{k}) + \xi(\vec{k},n,t), \tag{8}$$

where $a(t)=-\Gamma(r+u\langle\phi^2\rangle)$. The solution for $\phi_n(\vec{k},t)$ can now be written as

$$\phi_n(\vec{k},t) = e^{-\Gamma(k^2+n^2\pi^2/L^2)t+b(t)}\left(\int_0^t dt' e^{\Gamma(k^2+n^2\pi^2/L^2)t'-b(t')}\xi(\vec{k},t')\right) + \phi_n(\vec{k},0)e^{-\Gamma(k^2+n^2\pi^2/L^2)t+b(t)}, \tag{9}$$

where $b(t)=\int_0^t dt' a(t')$. The long time dynamics is dominated by the noise containing term and $\langle\phi^2\rangle$ in that limit is given by

$$\langle\phi^2\rangle = \frac{2\Gamma}{g(t)}\sum_n \int \frac{d^2\vec{k}}{(2\pi)^2}\int_0^t dt' e^{-2\Gamma(k^2+n^2\pi^2/L^2)(t-t')}g(t'), \tag{10}$$

where

$$g(t) = e^{-2b(t)}. \tag{11}$$

The dynamics of $g(t)$ is given by

$$\dot{g} = 2r\Gamma g + 4u\Gamma\int_0^t dt' g(t')\sum_n \int \frac{d^2\vec{k}}{(2\pi)^2}e^{-2\Gamma(k^2+n^2\pi^2/L^2)(t-t')}. \tag{12}$$

The critical point is now defined by the zero of the coefficient of the $k=0$, $n=1$ component of $\phi_n(\vec{k})$ in Eq. (8) and thus

$$r_c + u\langle\phi^2\rangle = -\frac{\pi^2}{L^2}. \tag{13}$$

We should point out here that we are not going to look at the dimensional crossover. If L becomes smaller than all other length scales in the problem, that is, $L\ll\xi$ (the correlation length at the three-dimensional transition temperature), the system would behave as a two-dimensional system which has no transition. r_c for such a system would tend to negative infinity as it indeed does if we let $L\to 0$ in Eq. (13).

We will, as we go along, define a dynamic length scale $\tilde{l}$ (or l_d in a different notation) and small L would mean $L\ll\tilde{l}$.

If we consider Eq. (12) at the critical point, then in the terms of the Laplace transform we arrive at

$$\tilde{g}(s) = 1\Big/\left(s + \frac{2\Gamma\pi^2}{L^2} + 4\Gamma^2 u(\Delta\tilde{J})\right), \tag{14}$$

where $\triangle\tilde{J}=\tilde{J}(0,L)-\tilde{J}(s,L)$ with

$$\tilde{J}(s,L) \equiv \sum_n \int \frac{d^2\vec{k}}{(2\pi)^2}\frac{1}{s+2\Gamma(k^2+n^2\pi^2/L^2)}. \tag{15}$$

We note that s allows us to introduce a length scale $\tilde{l}=(\Gamma/s)^{1/2}$. Since the long time behavior is of interest, our focus will be on small values of s, which will make $\tilde{l}$ a large length scale. The system size L will have to be compared to the length scale $\tilde{l}$. As explained in Appendix A, the small s form of $\tilde{g}(s)$ at finite values of L can be well approximated by

$$\tilde{g}(s) = \frac{2\pi}{u\Gamma}\left[\left(\frac{2\Gamma}{L^2 s}\right)^{1/2} + \frac{1}{2}\left(\frac{2\Gamma}{L^2 s}\right)\ln\left(\frac{L^2 s}{2\Gamma}\right)\right]. \tag{16}$$

The real time behavior is obtained by inverting the Laplace transform of $\tilde{g}(s)$ and we have

$$g(t) = \frac{C}{t^{1/2}}\left[1 + \frac{\sqrt{\pi}}{2}\left(\frac{2\Gamma t}{L^2}\right)^{1/2}\ln\left(\frac{L^2}{2\Gamma t}\right)\right]. \tag{17}$$

At this order the expressions for $a(t)$ and $b(t)$ become

$$2b(t) = \frac{1}{2}\ln t - \ln\left[1 + \frac{\sqrt{\pi}}{2}\left(\frac{2\Gamma t}{L^2}\right)^{1/2}\ln\left(\frac{L^2}{2\Gamma t}\right)\right]$$

and

$$a(t) = \frac{1}{4t} - \frac{\sqrt{\pi}}{8}\left(\frac{2\Gamma}{L^2 t}\right)^{1/2}\frac{\ln\left(\frac{L^2}{2e^2\Gamma t}\right)}{1 + \frac{\sqrt{\pi}}{2}\left(\frac{2\Gamma t}{L^2}\right)^{1/2}\ln\left(\frac{L^2}{2\Gamma t}\right)}. \tag{18}$$

We note that for L large enough so that $\frac{L^2}{\Gamma t}\gg 1$, $a(t)\simeq\frac{1}{4t}$, with the first correction given by

$$a(t) = \frac{1}{4t}\left[1 - \frac{\sqrt{\pi}}{2}\sqrt{\frac{2\Gamma t}{L^2}}\ln\left(\frac{L^2}{2e^2\Gamma t}\right)\right]. \tag{19}$$

For $\frac{\Gamma t}{L^2}\ll 1$, we can write $a(t)$ as $\frac{\epsilon(t)}{4t}$, where $\epsilon(t)$ is the quantity in brackets in Eq. (19) and is a slowly varying function in the range considered.

The global mode $\phi_1(0)$ now satisfies the equation of motion [see Eq. (8)]

$$\left(\frac{d}{dt} + \frac{\Gamma\pi^2}{L^2}\right)\phi_1(0,t) = \frac{\epsilon(t)}{4t}\phi_1(0,t) + \xi(t). \tag{20}$$

Under the transformation $\phi_1(0,t)=e^{-\Gamma t\pi^2/L^2}t^{\epsilon(t)/4}\psi(t)$ and making the slowly time varying approximation whereby

$(\dot{\epsilon}/\epsilon)t \ln t$ is considered significantly smaller than unity (that is $\Gamma t/L^2$ reasonably smaller than unity), we arrive at

$$\dot{\psi}(t) = e^{\Gamma t \pi^2/L^2} t^{-\epsilon(t)/4} \xi(t).$$

With the transformation of variable $\tau = t^x$, where x is a very slowly varying function of time in the range concerned and hence can be considered practically constant, we have

$$\dot{\psi}(\tau)\frac{d\tau}{dt} = e^{\Gamma t \pi^2/L^2} t^{-\epsilon(t)/4} \xi(t) = \tilde{f}(\tau). \tag{21}$$

The correlation function $\langle \tilde{f}(\tau)\tilde{f}(\tau')\rangle$ will be δ correlated in τ space provided

$$x = 1 - \frac{\epsilon}{2} - \frac{2\pi^2 \Gamma t}{L^2}\frac{1}{\ln(\Gamma t/L^2)}, \tag{22}$$

and Eq. (21) becomes

$$\dot{\psi}(\tau) = \tilde{f}(\tau). \tag{23}$$

Since the size-dependent correction in ϵ is $O(L^{-1})$, we can drop the last term to the leading order and write as the first effect of the finite size the relation

$$x = \frac{1}{2} + \frac{\sqrt{\pi}}{2}\sqrt{\frac{\Gamma t}{2L^2}} \ln\left(\frac{L^2}{e^2 \Gamma t}\right). \tag{24}$$

The persistence probability for the process of Eq. (23) goes as $\tau^{-1/2}$, and hence in the actual time variable t,

$$p(t) \sim \frac{1}{t^{1/4 + \sqrt{\pi \Gamma t/32 L^2}\, \ln(L^2/e^2 \Gamma t) + \cdots}}. \tag{25}$$

The decay is clearly hastened at a finite value of L. We would like to emphasize that the logarithmic corrections above is a special feature of $D=3$. Further, the $\frac{1}{4}$ in Eq. (25) is the spherical limit persistence exponent in $D=3$.

What happens is $L^2/\Gamma t$ becomes smaller than unity. Returning to Eq. (14) and Eq. (A3) (see Appendix A), it is now clear that the leading behavior of $g(t)$ is $e^{-\Gamma \pi^2 t/L^2}$ leading to $b(t) = \frac{\Gamma \pi^2}{L^2} t$ and $a(t) = \frac{\Gamma \pi^2}{L^2}$. This implies a dynamics

$$\frac{d}{dt}\phi_1(0,t) = -\frac{\Gamma \pi^2}{L^2}\phi_1(0,t) + \xi(t). \tag{26}$$

The associated $p(t)$ is known from Ref. [8] to be

$$p(t) \sim \sqrt{\frac{\Gamma \pi^2}{L^2}} \frac{e^{-\Gamma \pi^2 t/2L^2}}{\sqrt{\sinh\left(\frac{\Gamma \pi^2 t}{L^2}\right)}}. \tag{27}$$

A combination of the forms of Eq. (25) and Eq. (27) can be achieved by

$$p(t) \sim \frac{e^{-\Gamma \pi^2 t/2L^2}}{\left\{\frac{L^2}{\Gamma \pi^2}\sinh\left(\frac{\Gamma \pi^2 t}{L^2}\right)\right\}^{1/4+1/4+\alpha}}, \tag{28}$$

where

$$\alpha = \sqrt{\frac{32 L^2}{\pi \Gamma t}} \frac{1}{\ln\left(\frac{L^2}{e^2 \Gamma t}\right)}. \tag{29}$$

For $L^2 \gg \Gamma t$, we have the result of Majumdar *et al.* [1], that is, $p(t) \sim t^{-1/4}$, while for $L^2 \ll \Gamma t$, we regain Eq. (27).

Finally, we need to address what would happen if the spherical constraint is relaxed. We consider the equation of motion for $\langle \phi_i^2 \rangle$ with $i=1$ (say). In the spherical limit,

$$\frac{1}{2}\frac{d}{dt}\langle \phi_1^2\rangle = \Gamma \nabla^2 \langle \phi_1^2\rangle - \Gamma(r + u\langle \phi^2\rangle)\langle \phi_1^2\rangle + \langle \xi \phi_1\rangle. \tag{30}$$

If we consider the first correction to the spherical limit through the term $\frac{\Gamma u}{N}\langle [\Sigma_j \phi_j^2 - N\langle \phi^2\rangle]\phi_1^2\rangle$ then it is clear that the first nonvanishing term is $\frac{\Gamma u}{N}\Sigma_j \langle \phi_j \phi_1\rangle\langle \phi_j \phi_1\rangle = \frac{\Gamma u}{N}\langle \phi_1^2\rangle^2$. Thus the role of the term is to change u in Eq. (30) to $u(1+\frac{1}{N})$ (note that $\langle \phi^2\rangle = \langle \phi_1^2\rangle$). This implies a change of $O(1/N)$ in the persistence exponent θ and this will affect Eq. (29) to the extent that the $\frac{1}{4}$ in the exponent of the denominator of the right-hand side will acquire a $O(1/N)$ correction. This is a result that is independent of the spherical limit. If we imagine a perturbative calculation of $\langle \phi^2\rangle$ starting from Eq. (2) as would be done in an ϵ expansion, or in evaluating the next to leading term in N^{-1} expansion, the $sL^2/2\Gamma \ll 1$ limit always yields a small correction to the s obtained from time derivative. In this limit the growth rate is dominated by the small extension in the z direction and the collective effect of the low modes is small in comparison.

We note that the critical relaxation rate for a system governed by Eq. (2) goes as Γk^2, where k is the wave number of fluctuations. For finite size system the minimum value is of $O(L^{-1})$ and the lowest frequency is given by ΓL^{-2}. For any arbitrary time scale t, the relaxation rate allows us to define a dynamic length scale $l_d = \sqrt{\Gamma t}$. The ratio $L^2/\Gamma t$ which has featured so prominently in our discussion is thus the ratio L^2/l_d^2. The results of Majumdar *et al.* [1] are for the limit $L \gg l_d$. Our Eq. (28) is an attempt to capture the entire range from $L \gg l_d$ to $L \ll l_d$.

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APPENDIX A

In this appendix we focus on the evolution of $\triangle \tilde{J}$ and hence $\tilde{g}(s)$ for small values of s. This is predicated by our necessity of evaluating $g(t)$ at long times. The small values of s allows us to define a length scale $\tilde{l} = (\Gamma/s)^{1/2}$, which is big, and it is the competition between the system size L and $\tilde{l}$, which is of interest. We write

$$\Delta\tilde{J} = \tilde{J}(0,L) - \tilde{J}(s,L)$$

$$= \sum_{n\geq 1} \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{s}{s + 2\Gamma\left(k^2 + \frac{n^2\pi^2}{L^2}\right)} \frac{1}{2\Gamma\left(k^2 + \frac{n^2\pi^2}{L^2}\right)}$$

$$= \frac{1}{4\Gamma^2}\sum_n \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{s}{\frac{s}{2\Gamma} + \frac{n^2\pi^2}{L^2} + k^2} \frac{1}{k^2 + \frac{n^2\pi^2}{L^2}}$$

$$= \frac{\pi}{2\Gamma}\frac{1}{4\pi^2}\sum_n \ln\left(\frac{k^2 + \frac{n^2\pi^2}{L^2}}{k^2 + \frac{n^2\pi^2}{L^2} + \frac{s}{2\Gamma}}\right)\Bigg|_0^\infty$$

$$= \frac{1}{8\pi\Gamma}\sum_n \ln\left(1 + \frac{L^2 s}{2n^2\pi^2\Gamma}\right)$$

$$= \frac{1}{8\pi\Gamma} \ln\left(\frac{\sinh(\sqrt{x})}{\sqrt{x}}\right), \tag{A1}$$

with $x = sL^2/2\Gamma$.

We now need to explore the other limit $\tilde{l} \gg L$ as well as the limit $\tilde{l} \ll L$. The first limit corresponds to $\sqrt{x} \ll 1$ while the second limit corresponds to $\sqrt{x} \gg 1$. In the first case we arrive at

$$\Delta\tilde{J} = \frac{L^2 s}{96\pi\Gamma^2} \quad \text{for } sL^2/2\Gamma \ll 1. \tag{A2}$$

Since the denominator of Eq. (14) already contains a term linear in s, this limit of $\triangle\tilde{J}$ will not reveal any additional feature.

The second limit ($\sqrt{x} \gg 1$) gives us the first correction to $\triangle\tilde{J}$ and we have

$$\Delta\tilde{J} = \frac{1}{8\pi\Gamma}\sqrt{x}\left[1 - \frac{\ln x}{2\sqrt{x}}\right] = \frac{L}{8\pi\Gamma}\left(\frac{s}{2\Gamma}\right)^{1/2}\left[1 - \frac{\ln\left(\frac{L^2 s}{2\Gamma}\right)}{2L\left(\frac{s}{2\Gamma}\right)^{1/2}}\right]. \tag{A3}$$

Finally, since we are interested only in small values of s, term of $O(s)$ will be dropped in $\tilde{g}(s)$ of Eq. (14) hence with the help of Eq. (A3), we arrive at

$$\tilde{g}(s) = \frac{2\pi}{u\Gamma L(s/2\Gamma)^{1/2}}\left[1 + \frac{\ln(L^2 s/2\Gamma)}{2L(s/2\Gamma)^{1/2}}\right].$$

APPENDIX B

In this appendix we give the explicit calculations for $g(t)$, $b(t)$, and $a(t)$. Inverting Eq. (17) we have

$$g(t) = \frac{\sqrt{\pi}}{u\Gamma}\left(\frac{2\Gamma}{L^2 t}\right)^{1/2}\left[1 + \frac{\sqrt{\pi}}{2}\left(\frac{2\Gamma}{L^2 t}\right)^{1/2}\ln\left(\frac{L^2}{2\Gamma t}\right)\right].$$

Using Eq. (12) we have

$$2b(t) = -\ln g(t),$$

and therefore we have

$$2b(t) = \frac{1}{2}\ln t - \ln\left[1 + \frac{\sqrt{\pi}}{2}\left(\frac{2\Gamma t}{L^2}\right)^{1/2}\ln\left(\frac{L^2}{2\Gamma t}\right)\right].$$

This gives us for $a(t)$,

$$a(t) = \frac{db(t)}{dt} = \frac{1}{4t} - \frac{\sqrt{\pi}}{8}\left(\frac{2\Gamma}{L^2 t}\right)^{1/2}\frac{\ln\left(\frac{L^2}{2e^2\Gamma t}\right)}{1 + \frac{\sqrt{\pi}}{2}\left(\frac{2\Gamma t}{L^2}\right)^{1/2}\ln\left(\frac{L^2}{2\Gamma t}\right)}.$$

[1] S. N. Majumdar, A. J. Bray, S. J. Cornell, and C. Sire, Phys. Rev. Lett. **77**, 3704 (1996).

[2] B. Derrida, A. J. Bray, and C. Godrèche, J. Phys. A **27**, L357 (1994).

[3] D. Stauffer, J. Phys. A **27**, 5029 (1994).

[4] B. Derrida, V. Hakim, and V. Pasquier, Phys. Rev. Lett. **75**, 751 (1995).

[5] S. N. Majumdar, C. Sire, A. J. Bray, and S. J. Cornell, Phys. Rev. Lett. **77**, 2867 (1996).

[6] For a review of persistence invarious situations, see S. N. Majumdar, Curr. Sci. **77**, 370 (1999).

[7] B. Luki, S. Jeney, C. Tischer, A. J. Kulik, L. Forro, and E. L. Florin, Phys. Rev. Lett. **95**, 160601 (2005).

[8] D. Chakraborty and J. K. Bhattacharjee, Phys. Rev. E **75**, 011111 (2007).

[9] G. Manoj and P. Ray, Phys. Rev. E **62**, 7755 (2000).

[10] A. J. Bray and M. A. Moore, Phys. Rev. Lett. **38**, 735 (1977).

[11] V. Dohm, Phys. Scr., T **49**, 46 (1993).

[12] S. Bhattacharyya and J. K. Bhattacharya, Phys. Rev. B **59**, 3341 (1999).

[13] R. Paul, A. Gambassi, and G. Schehr, Europhys. Lett. **78**, 10007 (2007).